Introduction

1. Geometric figures. The part of space occupied by a physical object is called a **geometric solid**.

A geometric solid is separated from the surrounding space by a **surface**.

A part of the surface is separated from an adjacent part by a **line**.

A part of the line is separated from an adjacent part by a **point**.

The geometric solid, surface, line and point do not exist separately. However by way of abstraction we can consider a surface independently of the geometric solid, a line — independently of the surface, and the point — independently of the line. In doing so we should think of a surface as having no thickness, a line — as having neither thickness nor width, and a point — as having no length, no width, and no thickness.

A set of points, lines, surfaces, or solids positioned in a certain way in space is generally called a **geometric figure**. Geometric figures can move through space without change. Two geometric figures are called **congruent**, if by moving one of the figures it is possible to superimpose it onto the other so that the two figures become identified with each other in all their parts.

2. Geometry. A theory studying properties of geometric figures is called **geometry**, which translates from Greek as *land-measuring*. This name was given to the theory because the main purpose of geometry in antiquity was to measure distances and areas on the Earth's surface.

First concepts of geometry as well as their basic properties, are introduced as idealizations of the corresponding common notions and everyday experiences.

3. The plane. The most familiar of all surfaces is the flat surface, or the **plane**. The idea of the plane is conveyed by a window pane, or the water surface in a quiet pond.

We note the following property of the plane: *One can superimpose a plane on itself or any other plane in a way that takes one given point to any other given point, and this can also be done after flipping the plane upside down.*

4. The straight line. The most simple line is the **straight line**. The image of a thin thread stretched tight or a ray of light emitted through a small hole give an idea of what a straight line is. The following fundamental property of the straight line agrees well with these images:

For every two points in space, there is a straight line passing through them, and such a line is unique.

It follows from this property that:

If two straight lines are aligned with each other in such a way that two points of one line coincide with two points of the other, then the lines coincide in all their other points as well (because otherwise we would have two distinct straight lines passing through the same two points, which is impossible).

For the same reason, *two straight lines can intersect at most at one point.*

A straight line can lie in a plane. The following holds true:

If a straight line passes through two points of a plane, then all points of this line lie in this plane.

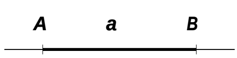

Figure 1

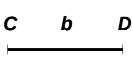

Figure 2

Figure 3

5. The unbounded straight line. Ray. Segment. Thinking of a straight line as extended indefinitely in both directions, one calls it an **infinite** (or **unbounded**) straight line.

A straight line is usually denoted by two uppercase letters marking any two points on it. One says "the line AB" or "BA" (Figure 1).

A part of the straight line bounded on both sides is called a **straight segment**. It is usually denoted by two letters marking its endpoints (the segment CD, Figure 2). Sometimes a straight line or a segment is denoted by one (lowercase) letter; one may say "the straight line a, the segment b."

Usually instead of "unbounded straight line" and "straight segment" we will simply say **line** and **segment** respectively.

Sometimes a straight line is considered which terminates in one direction only, for instance at the endpoint E (Figure 3). Such a straight line is called a **ray** (or **half-line**) drawn from E.

6. Congruent and non-congruent segments. *Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.* Suppose for example that we put the segment AB onto the segment CD (Figure 4) by placing the point A at the point C and aligning the ray AB with the ray CD. If, as a result of this, the points B and D merge, then the segments AB and CD are congruent. Otherwise they are not congruent, and the one which makes a part of the other is considered smaller.

Figure 4

To mark on a line a segment congruent to a given segment, one uses the **compass**, a drafting device which we assume familiar to the reader.

7. Sum of segments. The sum of several given segments (AB, CD, EF, Figure 5) is a segment which is obtained as follows. On a line, pick any point M and starting from it mark a segment MN congruent to AB, then mark the segments NP congruent to CD, and PQ congruent to EF, both going in the same direction as MN. Then the segment MQ will be the sum of the segments AB, CD and EF (which are called **summands** of this sum). One can similarly obtain the sum of any number of segments.

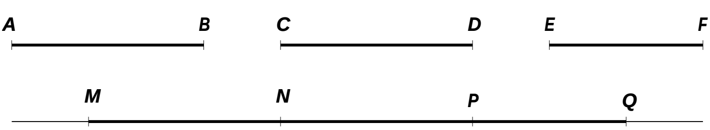

Figure 5

The sum of segments has the same properties as the sum of numbers. In particular it does not depend on the order of the summands (the **commutativity** law) and remains unchanged when some of the summands are replaced with their sum (the **associativity** law). For instance:

$$AB + CD + EF = AB + EF + CD = EF + CD + AB = \dots$$

and

$$AB + CD + EF = AB + (CD + EF) = CD + (AB + EF) = \dots.$$

8. Operations with segments. The concept of addition of segments gives rise to the concept of subtraction of segments, and multiplication and division of segments by a whole number. For example, the difference of AB and CD (if $AB > CD$) is a segment whose sum with CD is congruent to AB; the product of the segment AB with the number 3 is the sum of three segments each congruent to AB; the quotient of the segment AB by the number 3 is a third part of AB.

If given segments are measured by certain linear units (for instance, centimeters), and their lengths are expressed by the corresponding numbers, then the length of the sum of the segments is expressed by the sum of the numbers measuring these segments, the length of the difference is expressed by the difference of the numbers, etc.

9. The circle. If, setting the compass to an arbitrary step and, placing its pin leg at some point O of the plane (Figure 6), we begin to turn the compass around this point, then the other leg equipped with a pencil touching the plane will describe on the plane a continuous curved line all of whose points are the same distance away from O. This curved line is called a **circle**, and the point O — its **center**. A segment (OA, OB, OC in Figure 6) connecting the center with a point of the circle is called a **radius**. All radii of the same circle are congruent to each other.

Circles described by the compass set to the same radius are congruent because by placing their centers at the same point one will identify such circles with each other at all their points.

A line (MN, Figure 6) intersecting the circle at any two points is called a **secant**.

A segment (EF) both of whose endpoints lie on the circle is called a **chord**.

A chord (AD) passing through the center is called a **diameter**. A diameter is the sum of two radii, and therefore all diameters of the same circle are congruent to each other.

A part of a circle contained between any two points (for example, EmF) is called an **arc**.

The chord connecting the endpoints of an arc is said to **subtend** this arc.

An arc is sometimes denoted by the sign $\frown$; for instance, one writes: $\overset{\frown}{EmF}$.

The part of the plane bounded by a circle is called a **disk**.[2]

The part of a disk contained between two radii (the shaded part COB in Figure 6) is called a **sector**, and the part of the disk cut off by a secant (the part EmF) is called a **disk segment**.

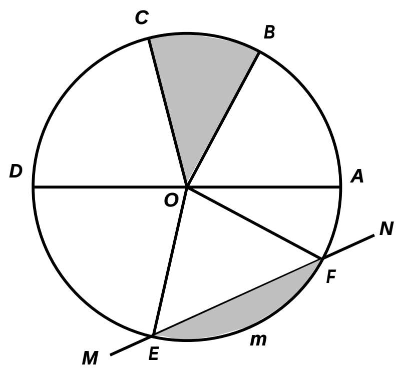

Figure 6

10. Congruent and non-congruent arcs. *Two arcs of the same circle (or of two congruent circles) are congruent if they can be aligned so that their endpoints coincide.* Indeed, suppose that we align the arc AB (Figure 7) with the arc CD by identifying the point A with the point C and directing the arc AB along the arc CD. If, as a result of this, the endpoints B and D coincide, then all the intermediate points of these arcs will coincide as well, since they are the same distance away from the center, and therefore $\overset{\frown}{AB}=\overset{\frown}{CD}$. But if B and D do not coincide, then the arcs are not congruent, and the one which is a part of the other is considered smaller.

11. Sum of arcs. The sum of several given arcs of the same radius is defined as an arc of that same radius which is composed from parts congruent respectively to the given arcs. Thus, pick an arbitrary point M (Figure 7) of the circle and mark the part MN

[2]Often the word "circle" is used instead of "disk." However one should avoid doing this since the use of the same term for different concepts may lead to mistakes.

congruent to AB. Next, moving in the same direction along the circle, mark the part NP congruent to CD. Then the arc MP will be the sum of the arcs AB and CD.

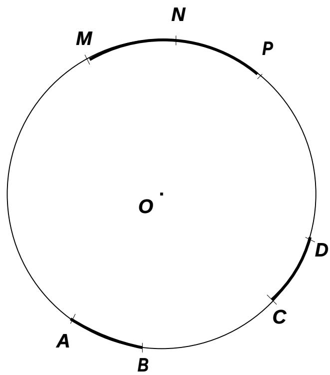

Figure 7

Adding arcs of the same radius one may encounter the situation when the sum of the arcs does not fit in the circle and one of the arcs partially covers another. In this case the sum will be an arc greater than the whole circle. For example, adding the arcs AmB and CnD (Figure 8) we obtain the arc consisting of the whole circle and the arc AD.

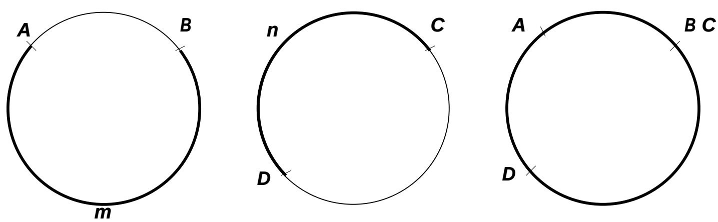

Figure 8

Similarly to addition of line segments, addition of arcs obeys the commutativity and associativity laws.

From the concept of addition of arcs one derives the concepts of subtraction of arcs, and multiplication and division of arcs by a whole number the same way as it was done for line segments.

12. Divisions of geometry. The subject of geometry can be divided into two parts: **plane geometry**, or **planimetry**, and **solid geometry**, or **stereometry**. Planimetry studies properties of those geometric figures all of whose elements fit the same plane.

EXERCISES

1. Give examples of geometric solids bounded by one, two, three, four planes (or parts of planes).

2. Show that if a geometric figure is congruent to another geometric figure, which is in its turn congruent to a third geometric figure, then the first geometric figure is congruent to the third.

3. Explain *why* two straight lines in space can intersect at most at one point.

4. Referring to §4, show that a plane not containing a given straight line can intersect it at most at one point.

5.⋆ [3] Give an example of a surface other than the plane which, like the plane, can be superimposed on itself in a way that takes any one given point to any other given point.
`Remark:` The required example is not unique.

6. Referring to §4, show that for any two points of a plane, there is a straight line lying *in this plane* and passing through them, and that such a line is unique.

7. Use a straightedge to draw a line passing through two points given on a sheet of paper. Figure out how to check that the line is really straight.
`Hint:` Flip the straightedge upside down.

8.⋆ Fold a sheet of paper and, using the previous problem, check that the edge is straight. Can you explain why the edge of a folded paper is straight?
`Remark:` There may exist several correct answers to this question.

9. Show that for each point lying in a plane there is a straight line lying in this plane and passing through this point. How many such lines are there?

10. Find surfaces other than the plane which, like the plane, together with each point lying on the surface contain a straight line passing through this point.
`Hint:` One can obtain such surfaces by bending a sheet of paper.

11. Referring to the definition of congruent figures given in §1, show that any two infinite straight lines are congruent; that any two rays are congruent.

12. On a given line, mark a segment congruent to four times a given segment, using a compass as few times as possible.

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[3] Stars ⋆ mark those exercises which we consider more difficult.

13. Is the sum (difference) of given segments unique? Give an example of two distinct segments which both are sums of the given segments. Show that these distinct segments are congruent.

14. Give an example of two non-congruent arcs whose endpoints coincide. Can such arcs belong to non-congruent circles? to congruent circles? to the same circle?

15. Give examples of non-congruent arcs subtended by congruent chords. Are there non-congruent chords subtending congruent arcs?

16. Describe explicitly the operations of subtraction of arcs, and multiplication and division of an arc by a whole number.

17. Follow the descriptions of operations with arcs, and show that multiplying a given arc by 3 and then dividing the result by 2, we obtain an arc congruent to the arc resulting from the same operations performed on the given arc in the reverse order.

18. Can sums (differences) of respectively congruent line segments, or arcs, be non-congruent? Can sums (differences) of respectively non-congruent segments, or arcs be congruent?

19. Following the definition of sum of segments or arcs, explain why addition of segments (or arcs) obeys the commutativity law.
`Hint:` Identify a segment (or arc) AB with BA.

Chapter 1

THE STRAIGHT LINE

1 Angles

13. Preliminary concepts. A figure formed by two rays drawn from the same point is called an **angle**. The rays which form the angle are called its **sides**, and their common endpoint is called the **vertex** of the angle. One should think of the sides as extending away from the vertex indefinitely.

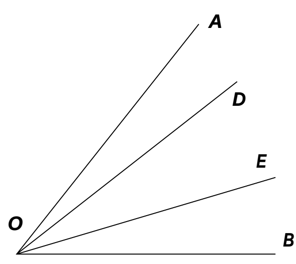

Figure 9

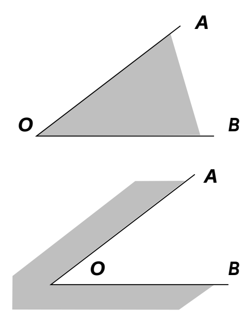

Figure 10

An angle is usually denoted by three uppercase letters of which the middle one marks the vertex, and the other two label a point on each of the sides. One says, e.g.: "the angle AOB" or "the angle BOA" (Figure 9). It is possible to denote an angle by one letter marking the vertex provided that no other angles with the same vertex are present on the diagram. Sometimes we will also denote an angle by a number placed inside the angle next to its vertex.

The sides of an angle divide the whole plane containing the angle into two regions. One of them is called the **interior** region of the angle, and the other is called the **exterior** one. Usually the interior region is considered the one that contains the segments joining any two points on the sides of the angle, e.g. the points A and B on the sides of the angle AOB (Figure 9). Sometimes however one needs to consider the other part of the plane as the interior one. In such cases a special comment will be made regarding which region of the plane is considered interior. Both cases are represented separately in Figure 10, where the interior region in each case is shaded.

Rays drawn from the vertex of an angle and lying in its interior (OD, OE, Figure 9) form new angles (AOD, DOE, EOB) which are considered to be parts of the angle (AOB).

In writing, the word "angle" is often replaced with the symbol $\angle$. For instance, instead of "angle AOB" one may write: $\angle AOB$.

14. Congruent and non-congruent angles. In accordance with the general definition of congruent figures (§1) *two angles are considered congruent if by moving one of them it is possible to identify it with the other.*

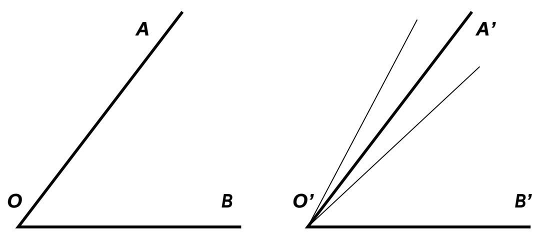

Figure 11

Suppose, for example, that we lay the angle AOB onto the angle $A'O'B'$ (Figure 11) in a way such that the vertex O coincides with O', the side OB goes along OB', and the interior regions of both angles lie on the same side of the line $O'B'$. If OA turns out to coincide with $O'A'$, then the angles are congruent. If OA turns out to lie inside or outside the angle $A'O'B'$, then the angles are non-congruent, and the one, that lies inside the other is said to be **smaller**.

15. Sum of angles. The sum of angles AOB and $A'O'B'$ (Figure 12) is an angle defined as follows. Construct an angle MNP congruent to the given angle AOB, and attach to it the angle PNQ, congruent to the given angle $A'O'B'$, as shown. Namely, the angle

MNP should have with the angle PNQ the same vertex N, a common side NP, and the interior regions of both angles should lie on the opposite sides of the common ray NP. Then the angle MNQ is called the sum of the angles AOB and $A'O'B'$. The interior region of the sum is considered the part of the plane comprised by the interior regions of the summands. This region contains the common side (NP) of the summands. One can similarly form the sum of three and more angles.

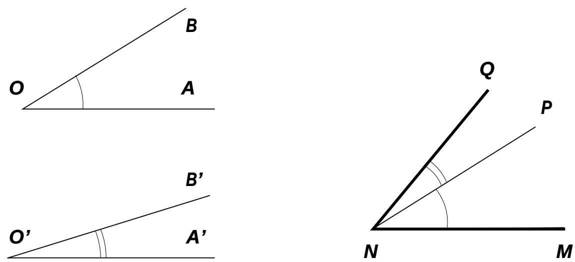

Figure 12

Addition of angles obeys the commutativity and associativity laws just the same way addition of segments does. From the concept of addition of angles one derives the concept of subtraction of angles, and multiplication and division of angles by a whole number.

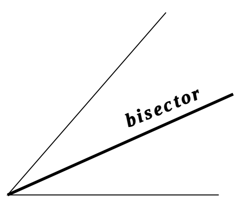

Figure 13

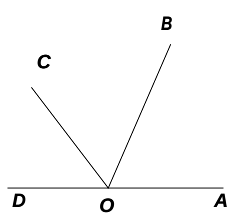

Figure 14

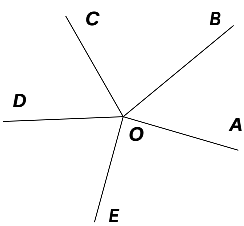

Figure 15

Very often one has to deal with the ray which divides a given angle into halves; this ray is called the **bisector** of the angle (Figure 13).

16. Extension of the concept of angle. When one computes the sum of angles some cases may occur which require special attention.

(1) It is possible that after addition of several angles, say, the

three angles: AOB, BOC and COD (Figure 14), the side OD of the angle COD will happen to be the continuation of the side OA of the angle AOB. We will obtain therefore the figure formed by two half-lines (OA and OD) drawn from the same point (O) and continuing each other. Such a figure is also considered an angle and is called a **straight angle**.

(2) It is possible that after the addition of several angles, say, the five angles: AOB, BOC, COD, DOE and EOA (Figure 15) the side OA of the angle EOA will happen to coincide with the side OA of the angle AOB. The figure formed by such rays (together with the whole plane surrounding the vertex O) is also considered an angle and is called a **full angle**.

(3) Finally, it is possible that added angles will not only fill in the whole plane around the common vertex, but will even overlap with each other, covering the plane around the common vertex for the second time, for the third time, and so on. Such an angle sum is congruent to one full angle added with another angle, or congruent to two full angles added with another angle, and so on.

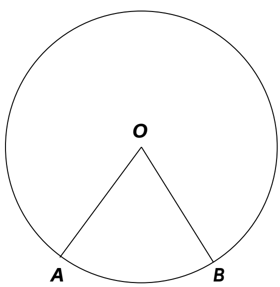

Figure 16

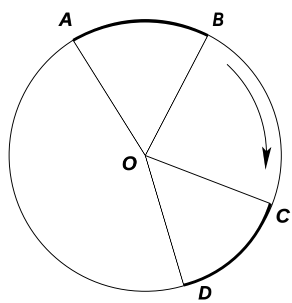

Figure 17

17. Central angle. The angle (AOB, Figure 16) formed by two radii of a circle is called a **central angle**; such an angle and the arc contained between the sides of this angle are said to *correspond* to each other.

Central angles and their corresponding arcs have the following properties.

In one circle, or two congruent circles:

(1) ***If central angles are congruent, then the corresponding arcs are congruent;***

(2) *Vice versa,* ***if the arcs are congruent, then the corre-***

sponding central angles are congruent.

Let $\angle AOB = \angle COD$ (Figure 17); we need to show that the arcs AB and CD are congruent too. Imagine that the sector AOB is rotated about the center O in the direction shown by the arrow until the radius OA coincides with OC. Then due to the congruence of the angles, the radius OB will coincide with OD; therefore the arcs AB and CD will coincide too, i.e. they are congruent.

The second property is established similarly.

18. Circular and angular degrees. Imagine that a circle is divided into 360 congruent parts and all the division points are connected with the center by radii. Then around the center, 360 central angles are formed which are congruent to each other as central angles corresponding to congruent arcs. Each of these arcs is called a **circular degree**, and each of those central angles is called an **angular degree**. Thus one can say that a circular degree is 1/360th part of the circle, and the angular degree is the central angle corresponding to it.

The degrees (both circular and angular) are further subdivided into 60 congruent parts called **minutes**, and the minutes are further subdivided into 60 congruent parts called **seconds**.

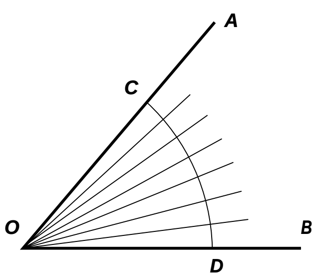

Figure 18

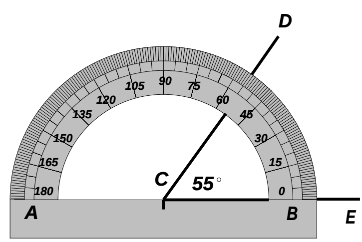

Figure 19

19. Correspondence between central angles and arcs. Let AOB be some angle (Figure 18). Between its sides, draw an arc CD of arbitrary radius with the center at the vertex O. Then the angle AOB will become the central angle corresponding to the arc CD. Suppose, for example, that this arc consists of 7 circular degrees (shown enlarged in Figure 18). Then the radii connecting the division points with the center obviously divide the angle AOB into 7 angular degrees. More generally, one can say that *an angle is measured by the arc corresponding to it*, meaning that an angle contains as many angular degrees, minutes and seconds as the corresponding arc contains circular degrees, minutes and seconds. For instance, if the arc CD contains 20 degrees 10 minutes and 15 seconds of circular units, then the angle AOB consists of 20 degrees 10 minutes and 15 seconds of angular units, which is customary to express as: $\angle AOB = 20^\circ 10' 15''$, using the symbols $^\circ$, $'$ and $''$ to denote degrees, minutes and seconds respectively.

Units of angular degree do not depend on the radius of the circle. Indeed, adding 360 angular degrees following the summation rule described in §15, we obtain the full angle at the center of the circle. Whatever the radius of the circle, this full angle will be the same. Thus one can say that an angular degree is 1/360th part of the full angle.

20. Protractor. This device (Figure 19) is used for measuring angles. It consists of a semi-disk whose arc is divided into 180°. To measure the angle DCE, one places the protractor onto the angle in a way such that the center of the semi-disk coincides with the vertex of the angle, and the radius CB lies on the side CE. Then the number of degrees in the arc contained between the sides of the angle DCE shows the measure of the angle. Using the protractor one can also draw an angle containing a given number of degrees (e.g. the angle of 90°, 45°, 30°, etc.).

EXERCISES

20. Draw any angle and, using a protractor and a straightedge, draw its bisector.

21. In the exterior of a given angle, draw another angle congruent to it. Can you do this in the interior of the given angle?

22. How many common sides can two distinct angles have?

23. Can two non-congruent angles contain 55 angular degrees each?

24. Can two non-congruent arcs contain 55 circular degrees each? What if these arcs have the same radius?

25. Two straight lines intersect at an angle containing 25°. Find the measures of the remaining three angles formed by these lines.

26. Three lines passing through the same point divide the plane into six angles. Two of them turned out to contain 25° and 55° respectively. Find the measures of the remaining four angles.

27.* Using only compass, construct a 1° arc on a circle, if a 19° arc of this circle is given.

2 Perpendicular lines

21. Right, acute and obtuse angles. An angle of 90° (congruent therefore to one half of the straight angle or to one quarter of the full angle) is called a **right angle**. An angle smaller than the right one is called **acute**, and a greater than right but smaller than straight is called **obtuse** (Figure 20).

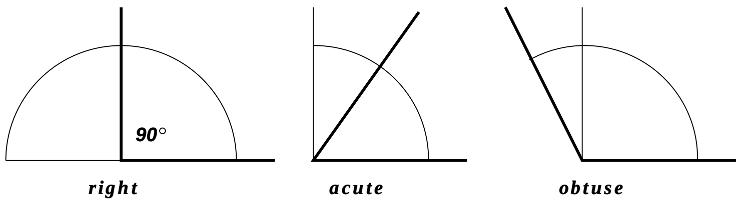

Figure 20

All right angles are, of course, *congruent to each other* since they contain the same number of degrees.

The measure of a right angle is sometimes denoted by d (the initial letter of the French word *droit* meaning "right").

22. Supplementary angles. Two angles (AOB and BOC, Figure 21) are called **supplementary** if they have one common side, and their remaining two sides form continuations of each other. Since the sum of such angles is a straight angle, *the sum of two supplementary angles is* 180° (in other words it is congruent to the sum of two right angles).

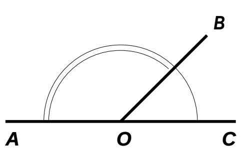

Figure 21

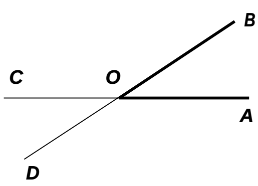

Figure 22

For each angle one can construct two supplementary angles. For example, for the angle AOB (Figure 22), prolonging the side AO we obtain one supplementary angle BOC, and prolonging the side BO we obtain another supplementary angle AOD. *Two angles supplementary to the same one are congruent to each other*, since they both contain the same number of degrees, namely the number that supplements the number of degrees in the angle AOB to 180° contained in a straight angle.

If AOB is a right angle (Figure 23), i.e. if it contains 90°, then each of its supplementary angles COB and AOD must also be right, since it contains 180° − 90°, i.e. 90°. The fourth angle COD has to be right as well, since the three angles AOB, BOC and AOD contain 270° altogether, and therefore what is left from 360° for the fourth angle COD is 90° too. Thus, *if one of the four angles formed by two intersecting lines* (AC and BD, Figure 23) *is right, then the other three angles must be right as well.*

23. A perpendicular and a slant. In the case when two supplementary angles are not congruent to each other, their common side (OB, Figure 24) is called a **slant** [1] to the line (AC) containing the other two sides. When, however, the supplementary angles are congruent (Figure 25) and when, therefore, each of the angles is right, the common side is called a **perpendicular** to the line containing the other two sides. The common vertex (O) is called the **foot of the slant** in the first case, and the **foot of the perpendicular** in the second.

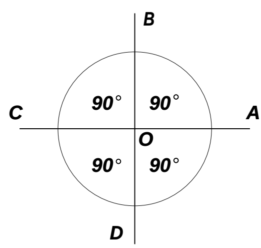

Figure 23

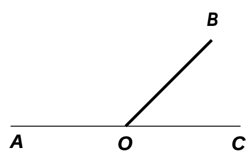

Figure 24

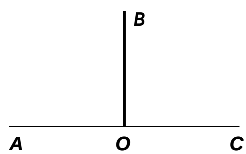

Figure 25

Two lines (AC and BD, Figure 23) intersecting at a right angle are called **perpendicular** to each other. The fact that the line AC is perpendicular to the line BD is written: $AC \perp BD$.

`Remarks.` (1) If a perpendicular to a line AC (Figure 25) needs to be drawn through a point O lying on this line, then the perpendicular is said to be "erected" to the line AC, and if the perpendicular needs to be drawn through a point B lying outside the line, then the perpendicular is said to be "dropped" to the line (no matter if it is upward, downward or sideways).

[1] Another name used for a slant is an **oblique line**.

(2) Obviously, at any given point of a given line, on either side of it, one can erect a perpendicular, and such a perpendicular is unique.

24. Let us prove that ***from any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.***

Let a line AB (Figure 26) and an arbitrary point M outside the line be given. We need to show that, first, one can drop a perpendicular from this point to AB, and second, that there is only one such perpendicular.

Imagine that the diagram is folded so that the upper part of it is identified with the lower part. Then the point M will take some position N. Mark this position, unfold the diagram to the initial form and then connect the points M and N by a line. Let us show now that the resulting line MN is perpendicular to AB, and that any other line passing through M, for example MD, is not perpendicular to AB. For this, fold the diagram again. Then the point M will merge with N again, and the points C and D will remain in their places. Therefore the line MC will be identified with NC, and MD with ND. It follows that $\angle MCB = \angle BCN$ and $\angle MDC = \angle CDN$.

But the angles MCB and BCN are supplementary. Therefore each of them is right, and hence $MN \perp AB$. Since MDN is not a straight line (because there can be no two straight lines connecting the points M and N), then the sum of the two congruent angles MDC and CDN is not equal to $2d$. Therefore the angle MDC is not right, and hence MD is not perpendicular to AB. Thus one can drop no other perpendicular from the point M to the line AB.

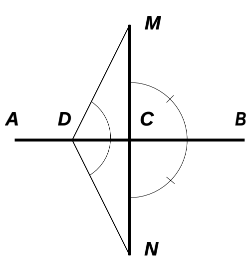

Figure 26

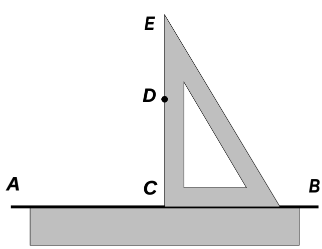

Figure 27

25. The drafting triangle. For practical construction of a perpendicular to a given line it is convenient to use a **drafting triangle** made to have one of its angles right. To draw the perpendicular to a line AB (Figure 27) through a point C lying on this line, or through a point D taken outside of this line, one can align a straightedge with the line AB, the drafting triangle with the straightedge, and then slide the triangle along the straightedge until the other side of the right angle hits the point C or D, and then draw the line CE.

26. Vertical angles. Two angles are called **vertical** if the sides of one of them form continuations of the sides of the other. For instance, at the intersection of two lines AB and CD (Figure 28) two pairs of vertical angles are formed: AOD and COB, AOC and DOB (and four pairs of supplementary angles).

Two vertical angles are congruent to each other (for example, $\angle AOD = \angle BOC$) since each of them is supplementary to the same angle (to $\angle DOB$ or to $\angle AOC$), and such angles, as we have seen (§22), are congruent to each other.

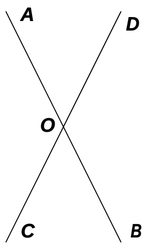

Figure 28

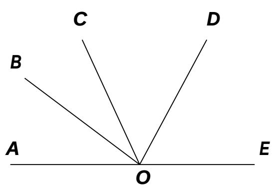

Figure 29

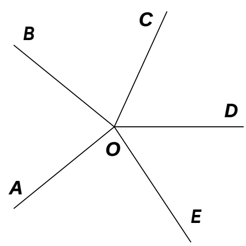

Figure 30

27. Angles that have a common vertex. It is useful to remember the following simple facts about angles that have a common vertex:

(1) *If the sum of several angles* (AOB, BOC, COD, DOE, Figure 29) *that have a common vertex is congruent to a straight angle, then the sum is* $2d$*, i.e.* $180°$*.*

(2) *If the sum of several angles* (AOB, BOC, COD, DOE, EOA, Figure 30) *that have a common vertex is congruent to the full angle, then it is* $4d$*, i.e.* $360°$*.*

(3) *If two angles* (AOB and BOC, Figure 24) *have a common vertex* (O) *and a common side* (OB) *and add up to* $2d$ (i.e. $180°$), *then their two other sides* (AO and OC) *form continuations of each other* (i.e. such angles are supplementary).

EXERCISES

28. Is the sum of the angles $14°24'44''$ and $75°35'25''$ acute or obtuse?

29. Five rays drawn from the same point divide the full angle into five congruent parts. How many different angles do these five rays form? Which of these angles are congruent to each other? Which of them are acute? Obtuse? Find the degree measure of each of them.

30. Can both angles, whose sum is the straight angle, be acute? obtuse?

31. Find the smallest number of acute (or obtuse) angles which add up to the full angle.

32. An angle measures $38°20'$; find the measure of its supplementary angles.

33. One of the angles formed by two intersecting lines is $2d/5$. Find the measures of the other three.

34. Find the measure of an angle which is congruent to twice its supplementary one.

35. Two angles ABC and CBD having the common vertex B and the common side BC are positioned in such a way that they do not cover one another. The angle $ABC = 100°20'$, and the angle $CBD = 79°40'$. Do the sides AB and BD form a straight line or a bent one?

36. Two distinct rays, perpendicular to a given line, are erected at a given point. Find the measure of the angle between these rays.

37. In the interior of an obtuse angle, two perpendiculars to its sides are erected at the vertex. Find the measure of the obtuse angle, if the angle between the perpendiculars is $4d/5$.

`Prove:`

38. Bisectors of two supplementary angles are perpendicular to each other.

39. Bisectors of two vertical angles are continuations of each other.

40. If at a point O of the line AB (Figure 28) two congruent angles AOD and BOC are built on the opposites sides of AB, then their sides OD and OC form a straight line.

41. If from the point O (Figure 28) rays OA, OB, OC and OD are constructed in such a way that $\angle AOC = \angle DOB$ and $\angle AOD = \angle COB$, then OB is the continuation of OA, and OD is the continuation of OC.
`Hint:` Apply §27, statements 2 and 3.

3 Mathematical propositions

28. Theorems, axioms, definitions. From what we have said so far one can conclude that some geometric statements we consider quite obvious (for example, the properties of planes and lines in §3 and §4) while some others are established by way of reasoning (for example, the properties of supplementary angles in §22 and vertical angles in §26). In geometry, this process of reasoning is a principal way to discover properties of geometric figures. It would be instructive therefore to acquaint yourself with the forms of reasoning usual in geometry.

All facts established in geometry are expressed in the form of propositions. These propositions are divided into the following types.

Definitions. Definitions are propositions which explain what meaning one attributes to a name or expression. For instance, we have already encountered the definitions of central angle, right angle, perpendicular lines, etc.

Axioms. Axioms [2] are those facts which are accepted without proof. This includes, for example, some propositions we encountered earlier (§4): through any two points there is a unique line; if two points of a line lie in a given plane then all points of this line lie in the same plane.

Let us also mention the following axioms which apply to any kind of quantities:

if each of two quantities is equal to a third quantity, then these two quantities are equal to each other;

if the same quantity is added to or subtracted from equal quantities, then the equality remains true;

if the same quantity is added to or subtracted from unequal quantities, then the inequality remains unchanged, i.e. the greater quantity remains greater.

Theorems. Theorems are those propositions whose truth is found only through a certain reasoning process (proof). The following propositions may serve as examples:

if in one circle or two congruent circles some central angles are congruent, then the corresponding arcs are congruent;

if one of the four angles formed by two intersecting lines turns out to be right, then the remaining three angles are right as well.

[2] In geometry, some axioms are traditionally called **postulates**.

Corollaries. Corollaries are those propositions which follow directly from an axiom or a theorem. For instance, it follows from the axiom "there is only one line passing through two points" that "two lines can intersect at one point at most."

29. The content of a theorem. In any theorem one can distinguish two parts: the hypothesis and the conclusion. The **hypothesis** expresses what is considered given, the **conclusion** what is required to prove. For example, in the theorem "if central angles are congruent, then the corresponding arcs are congruent" the hypothesis is the first part of the theorem: "if central angles are congruent," and the conclusion is the second part: "then the corresponding arcs are congruent;" in other words, it is given (known to us) that the central angles are congruent, and it is required to prove that under this hypothesis the corresponding arcs are congruent.

The hypothesis and the conclusion of a theorem may sometimes consist of several separate hypotheses and conclusions; for instance, in the theorem "if a number is divisible by 2 and by 3, then it is divisible by 6," the hypothesis consists of two parts: "if a number is divisible by 2" and "if the number is divisible by 3."

It is useful to notice that any theorem can be rephrased in such a way that the hypothesis will begin with the word "if," and the conclusion with the word "then." For example, the theorem "vertical angles are congruent" can be rephrased this way: "*if* two angles are vertical, *then* they are congruent."

30. The converse theorem. The theorem converse to a given theorem is obtained by replacing the hypothesis of the given theorem with the conclusion (or some part of the conclusion), and the conclusion with the hypothesis (or some part of the hypothesis) of the given theorem. For instance, the following two theorems are converse to each other:

If central angles are congruent, then the corresponding arcs are congruent.

If arcs are congruent, then the corresponding central angles are congruent.

If we call one of these theorems **direct**, then the other one should be called **converse**.

In this example both theorems, the direct and the converse one, turn out to be true. This is not always the case. For example the theorem: "if two angles are vertical, then they are congruent" is true, but the converse statement: "if two angles are congruent, then they are vertical" is false.

Indeed, suppose that in some angle the bisector is drawn (Figure 13). It divides the angle into two smaller ones. These smaller angles are congruent to each other, but they are not vertical.

EXERCISES

42. Formulate definitions of supplementary angles (§22) and vertical angles (§26) using the notion of *sides* of an angle.

43. Find in the text the definitions of an angle, its vertex and sides, in terms of the notion of a *ray drawn from a point.*

44.* In Introduction, find the definitions of a ray and a straight segment in terms of the notions of a *straight line* and a point. Are there definitions of a point, line, plane, surface, geometric solid? Why? `Remark:` These are examples of geometric notions which are considered **undefinable**.

45. Is the following proposition from §6 a definition, axiom or theorem: "Two segments are congruent if they can be laid one onto the other so that their endpoints coincide"?

46. In the text, find the definitions of a geometric figure, and congruent geometric figures. Are there definitions of congruent segments, congruent arcs, congruent angles? Why?

47. Define a circle.

48. Formulate the proposition converse to the theorem: "If a number is divisible by 2 and by 3, then it is divisible by 6." Is the converse proposition true? Why?

49. In the proposition from §10: "Two arcs of the same circle are congruent if they can be aligned so that their endpoints coincide," separate the hypothesis from the conclusion, and state the converse proposition. Is the converse proposition true? Why?

50. In the theorem: "Bisectors of supplementary angles are perpendicular," separate the hypothesis from the conclusion, and formulate the converse proposition. Is the converse proposition true?

51. Give an example that disproves the proposition: "If the bisectors of two angles with a common vertex are perpendicular, then the angles are supplementary." Is the converse proposition true?

4 Polygons and triangles

31. Broken lines. Straight segments not lying on the same line are said to form a **broken line** (Figures 31, 32) if the endpoint of the first segment is the beginning of the second one, the endpoint of the second segment is the beginning of the third one, and so on. These segments are called **sides**, and the vertices of the angles formed by the adjacent segments **vertices** of the broken line. A broken line is denoted by the row of letters labeling its vertices and endpoints; for instance, one says: "the broken line $ABCDE$."

A broken line is called **convex** if it lies on one side of each of its segments continued indefinitely in both directions. For example, the broken line shown in Figure 31 is convex while the one shown in Figure 32 is not (it lies not on one side of the line BC).

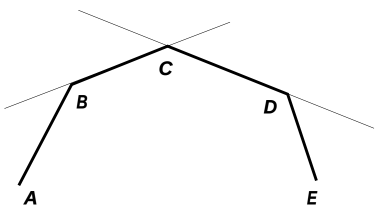

Figure 31

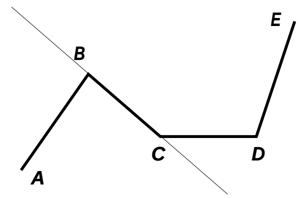

Figure 32

A broken line whose endpoints coincide is called **closed** (e.g. the lines $ABCDE$ or $ADCBE$ in Figure 33). A closed broken line may have self-intersections. For instance, in Figure 33, the line $ADCBE$ is self-intersecting, while $ABCDE$ is not.

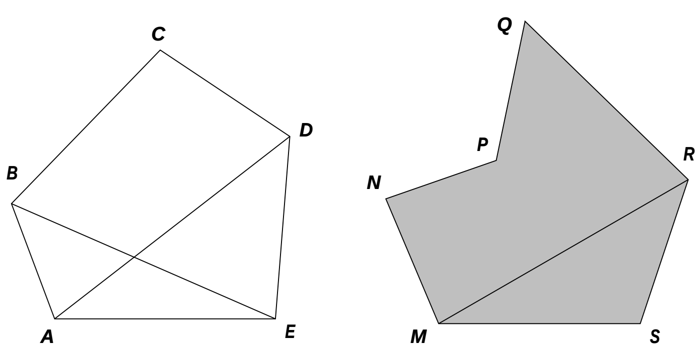

Figure 33

32. Polygons. The figure formed by a non-self-intersecting closed broken line together with the part of the plane bounded by this line is called a **polygon** (Figure 33). The sides and vertices of this broken line are called respectively **sides** and **vertices** of the polygon, and the angles formed by each two adjacent sides **(interior) angles** of the polygon. More precisely, the interior of a polygon's angle is considered that side which contains the interior part of the polygon in the vicinity of the vertex. For instance, the angle at the vertex P of the polygon $MNPQRS$ is the angle greater than $2d$ (with the interior region shaded in Figure 33). The broken line itself is called the **boundary** of the polygon, and the segment congruent to the sum of all of its sides — the **perimeter**. A half of the perimeter is often referred to as the **semiperimeter**.

A polygon is called **convex** if it is bounded by a convex broken line. For example, the polygon $ABCDE$ shown in Figure 33 is convex while the polygon $MNPQRS$ is not. We will mainly consider convex polygons.

Any segment (like AD, BE, MR, ..., Figure 33) which connects two vertices not belonging to the same side of a polygon is called a **diagonal** of the polygon.

The smallest number of sides in a polygon is three. Polygons are named according to the number of their sides: **triangles**, **quadrilaterals**, **pentagons**, **hexagons**, and so on.

The word "triangle" will often be replaced by the symbol $\triangle$.

33. Types of triangles. Triangles are classified by relative lengths of their sides and by the magnitude of their angles. With respect to the lengths of sides, triangles can be **scalene** (Figure 34) — when all three sides have different lengths, **isosceles** (Figure 35) — when two sides are congruent, and **equilateral** (Figure 36) — when all three sides are congruent.

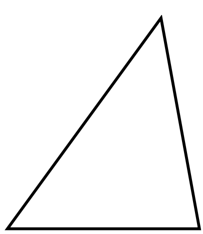

Figure 34

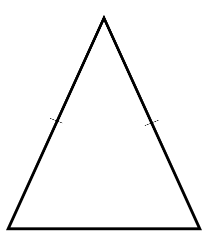

Figure 35

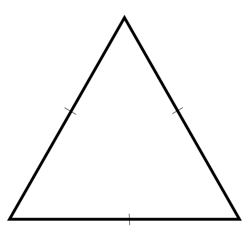

Figure 36

With respect to the magnitude of angles, triangles can be **acute** (Figure 34) — when all three angles are acute, **right** (Figure 37) —

when among the angles there is a right one, and **obtuse** (Figure 38) — when among the angles there is an obtuse one. [3]

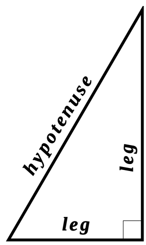

Figure 37

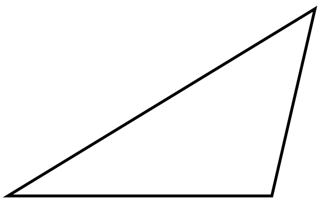

Figure 38

In a right triangle, the sides of the right angle are called **legs**, and the side opposite to the right angle the **hypotenuse**.

34. Important lines in a triangle. One of a triangle's sides is often referred to as **the base**, in which case the opposite vertex is called *the* vertex of the triangle, and the other two sides are called **lateral**. Then the perpendicular dropped from the vertex to the base or to its continuation is called an **altitude**. Thus, if in the triangle ABC (Figure 39), the side AC is taken for the base, then B is the vertex, and BD is the altitude.

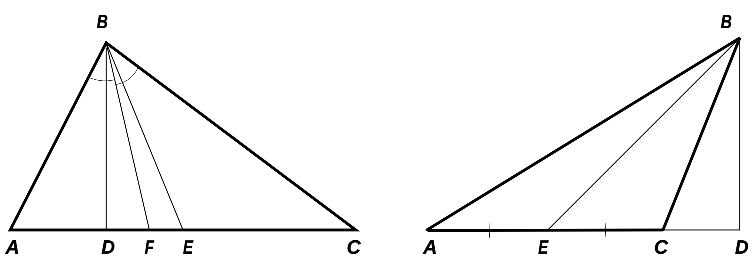

Figure 39

The segment (BE, Figure 39) connecting the vertex of a triangle with the midpoint of the base is called a **median**. The segment (BF) dividing the angle at the vertex into halves is called a **bisector** of the triangle (which generally speaking differs from both the median and the altitude).

[3] We will see in §43 that a triangle may have at most one right or obtuse angle.

Any triangle has three altitudes, three medians, and three bisectors, since each side of the triangle can take on the role of the base.

In an isosceles triangle, usually the side other than each of the two congruent ones is called the base. Respectively, the vertex of an isosceles triangle is the vertex of that angle which is formed by the congruent sides.

EXERCISES

52. Four points on the plane are vertices of three different quadrilaterals. How can this happen?

53. Can a convex broken line self-intersect?

54. Is it possible to tile the entire plane by non-overlapping polygons all of whose angles contain 140° each?

55. Prove that each diagonal of a quadrilateral either lies entirely in its interior, or entirely in its exterior. Give an example of a pentagon for which this is false.

56. Prove that a closed convex broken line is the boundary of a polygon.

57. Is an equilateral triangle considered isosceles? Is an isosceles triangle considered scalene?

58.* How many intersection points can three straight lines have?

59. Prove that in a right triangle, three altitudes pass through a common point.

60. Show that in any triangle, every two medians intersect. Is the same true for every two bisectors? altitudes?

61. Give an example of a triangle such that only one of its altitudes lies in its interior.

5 Isosceles triangles and symmetry

35. `Theorems.`

(1) ***In an isosceles triangle, the bisector of the angle at the vertex is at the same time the median and the altitude.***

(2) ***In an isosceles triangle, the angles at the base are congruent.***

Let $\triangle ABC$ (Figure 40) be isosceles, and let the line BD be the bisector of the angle B at the vertex of the triangle. It is required to prove that this bisector BD is also the median and the altitude.

Imagine that the diagram is folded along the line BD so that $\angle ABD$ falls onto $\angle CBD$. Then, due to congruence of the angles 1 and 2, the side AB will fall onto the side CB, and due to congruence of these sides, the point A will merge with C. Therefore DA will coincide with DC, the angle 3 will coincide with the angle 4, and the angle 5 with 6. Therefore

$$DA = DC, \quad \angle 3 = \angle 4, \quad \text{and} \quad \angle 5 = \angle 6.$$

It follows from $DA = DC$ that BD is the median. It follows from the congruence of the angles 3 and 4 that these angles are right, and hence BD is the altitude of the triangle. Finally, the angles 5 and 6 at the base of the triangle are congruent.

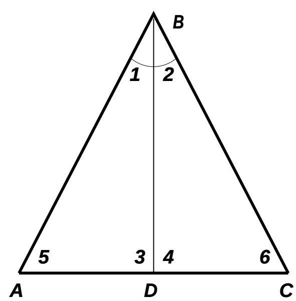

Figure 40

36. `Corollary.` We see that in the isosceles triangle ABC (Figure 40) the very same line BD possesses four properties: it is the bisector drawn from the vertex, the median to the base, the altitude dropped from the vertex to the base, and finally the perpendicular erected from the base at its midpoint.

Since each of these properties determines the position of the line BD unambiguously, then the validity of any of them implies all the others. For example, *the altitude dropped to the base of an isosceles triangle is at the same time its bisector drawn from the vertex, the median to the base, and the perpendicular erected at its midpoint.*

37. Axial symmetry. If two points (A and A', Figure 41) are situated on the opposite sides of a line a, on the same perpendicular to this line, and the same distance away from the foot of the perpendicular (i.e. if AF is congruent to FA'), then such points are called **symmetric** about the line a.

Two figures (or two parts of the same figure) are called symmetric about a line if for each point of one figure (A, B, C, D, E, ..., Figure 41) the point symmetric to it about this line (A', B', C', D', E', ...) belongs to the other figure, and *vice versa.* A figure is said to have an **axis of symmetry** a if this figure is symmetric to itself about the line a, i.e. if for any point of the figure the symmetric point also belongs to the figure.

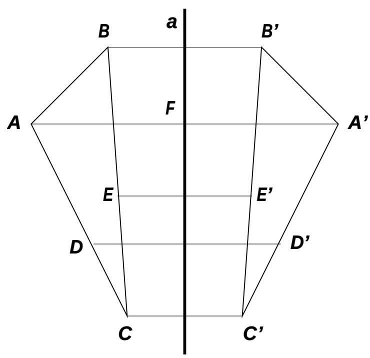

Figure 41

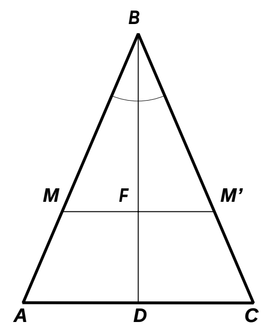

Figure 42

For example, we have seen that the isosceles triangle ABC (Figure 42) is divided by the bisector BD into two triangles (left and right) which can be identified with each other by folding the diagram along the bisector. One can conclude from this that whatever point is taken on the left half of the isosceles triangle, one can always find the point symmetric to it in the right half. For instance, on the side AB, take a point M. Mark on the side BC the segment BM' congruent to BM. We obtain the point M' in the triangle symmetric to M about the axis BD. Indeed, $\triangle MBM'$ is isosceles since $BM = BM'$. Let F denote the intersection point of the segment MM' with the bisector BD of the angle B. Then BK is the bisector in the isosceles triangle MBM'. By §35 it is also the altitude and the median. Therefore MM' is perpendicular to BD, and $MF = M'F$, i.e. M and M' are situated on the opposite sides of BD, on the same perpendicular to BD, and the same distance away from its foot F. Thus ***in an isosceles triangle, the bisector of the angle at the vertex is an axis of symmetry of the triangle.***

38. Remarks. (1) Two symmetric figures can be superimposed by rotating one of them in space about the axis of symmetry until the rotated figure falls into the original plane again. Conversely, if two figures can be identified with each other by turning the plane in space about a line lying in the plane, then these two figures are symmetric about this line.

(2) Although symmetric figures can be superimposed, they are not identical in their position in the plane. This should be understood in the following sense: in order to superimpose two symmetric figures it is *necessary* to flip one of them around and therefore to pull it off the plane temporarily; if however a figure is bound to remain in the plane, no motion can generally speaking identify it with the figure symmetric to it about a line. For example, Figure 43 shows two pairs of symmetric letters: "b" and "d," and "p" and "q." By rotating the letters inside the page one can transform "b" into "q," and "d" into "p," but it is impossible to identify "b" or "q" with "d" or "p" without lifting the symbols off the page.

(3) Axial symmetry is frequently found in nature (Figure 44).

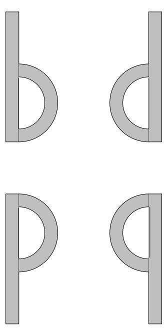

Figure 43

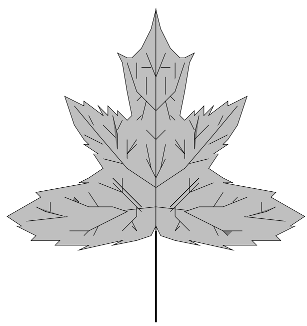

Figure 44

EXERCISES

62. How many axes of symmetry does an equilateral triangle have? How about an isosceles triangle which is not equilateral?

63.* How many axes of symmetry can a quadrilateral have?

64. A **kite** is a quadrilateral symmetric about a diagonal. Give an example of: (a) a kite; (b) a quadrilateral which is not a kite but has an axis of symmetry.

65. Can a pentagon have an axis of symmetry passing through two (one, none) of its vertices?

66.* Two points A and B are given on the same side of a line MN.

Find a point C on MN such that the line MN would make congruent angles with the sides of the broken line ACB.

`Prove theorems:`

67. In an isosceles triangle, two medians are congruent, two bisectors are congruent, two altitudes are congruent.

68. If from the midpoint of each of the congruent sides of an isosceles triangle, the segment perpendicular to this side is erected and continued to its intersection with the other of the congruent sides of the triangle, then these two segments are congruent.

69. A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

70. An equilateral triangle is **equiangular** (i.e. all of its angles are congruent).

71. Vertical angles are symmetric to each other with respect to the bisector of their supplementary angles.

72. A triangle that has two axes of symmetry has three axes of symmetry.

73. A quadrilateral is a kite if it has an axis of symmetry passing through a vertex.

74. Diagonals of a kite are perpendicular.

6 Congruence tests for triangles

39. Preliminaries. As we know, two geometric figures are called congruent if they can be identified with each other by superimposing. Of course, in the identified triangles, all their corresponding elements, such as sides, angles, altitudes, medians and bisectors, are congruent. However, in order to ascertain that two triangles are congruent, there is no need to establish congruence of all their corresponding elements. It suffices only to verify congruence of some of them.

40. `Theorems.` [4]

(1) **SAS-test**: ***If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.***

(2) **ASA-test**: ***If one side and two angles adjacent to it in one triangle are congruent respectively to one side and two***

[4]SAS stands for “side–angle–side”, ASA for “angle-side-angle, and of course SSS for “side-side-side.”

angles adjacent to it in another triangle, then such triangles are congruent.

(3) **SSS-test**: ***If three sides of one triangle are congruent respectively to three sides of another triangle, then such triangles are congruent.***

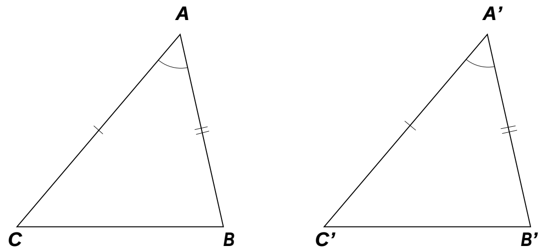

Figure 45

(1) Let ABC and $A'B'C'$ be two triangles (Figure 45) such that

$$AC = A'C', \quad AB = A'B', \quad \angle A = \angle A'.$$

It is required to prove that these triangles are congruent.

Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that A would coincide with A', the side AC would go along $A'C'$, and the side AB would lie on the same side of $A'C'$ as $A'B'$. [5] Then: since AC is congruent to $A'C'$, the point C will merge with C'; due to congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$, and due to congruence of these sides, the point B will merge with B'. Therefore the side BC will coincide with $B'C'$ (since two points can be joined by only one line), and hence the entire triangles will be identified with each other. Thus they are congruent.

(2) Let ABC and $A'B'C'$ (Figure 46) be two triangles such that

$$\angle C = \angle C', \quad \angle B = \angle B', \quad CB = C'B'.$$

It is required to prove that these triangles are congruent. Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that the point C would coincide with C', the side CB would go along $C'B'$, and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of

[5] For this and some other operations in this section it might be necessary to flip the triangle over.

the angles B and B', and C and C', the side BA will go along $B'A'$, and the side CA will go along $C'A'$. Since two lines can intersect only at one point, the vertex A will have to merge with A'. Thus the triangles are identified and are therefore congruent.

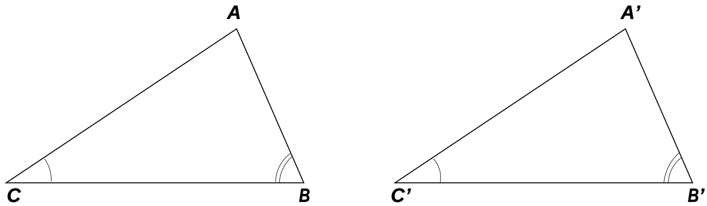

Figure 46

(3) Let ABC and $A'B'C'$ be two triangles such that

$$AB = A'B', \quad BC = B'C', \quad CA = C'A'.$$

It is required to prove that these triangles are congruent. Proving this test by superimposing, the same way as we proved the first two tests, turns out to be awkward, because knowing nothing about the measure of the angles, we would not be able to conclude from coincidence of two corresponding sides that the other sides coincide as well. Instead of superimposing, let us apply *juxtaposing*.

Juxtapose $\triangle ABC$ and $\triangle A'B'C'$ in such a way that their congruent sides AC and $A'C'$ would coincide (i.e. A would merge with A' and C with C'), and the vertices B and B' would lie on the opposite sides of $A'C'$. Then $\triangle ABC$ will occupy the position $\triangle A'B''C'$ (Figure 47). Joining the vertices B' and B'' we obtain two isosceles triangles $B'A'B''$ and $B'C'B''$ with the common base $B'B''$. But in an isosceles triangle, the angles at the base are congruent (§35). Therefore $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, and hence $\angle A'B'C' = \angle A'B''C' = \angle B$. But then the given triangles must be congruent, since two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in the other triangle.

`Remark.` In congruent triangles, congruent angles are opposed to congruent sides, and conversely, congruent sides are opposed to congruent angles.

The congruence tests just proved, and the skill of recognizing congruent triangles by the above criteria facilitate solutions to many geometry problems and are necessary in the proofs of many theorems. These congruence tests are the principal means in discovering properties of complex geometric figures. The reader will have many occasions to see this.

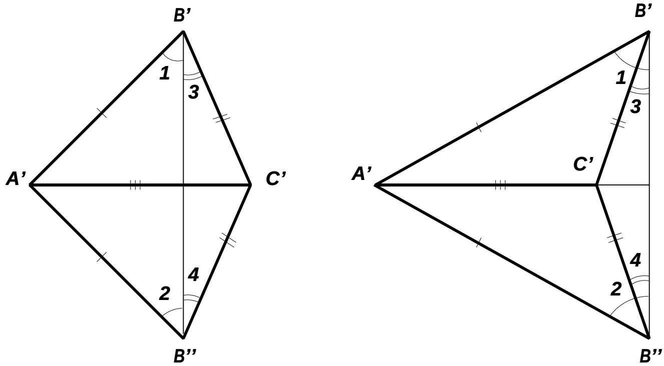

Figure 47

EXERCISES

75. Prove that a triangle that has two congruent angles is isosceles.

76. In a given triangle, an altitude is a bisector. Prove that the triangle is isosceles.

77. In a given triangle, an altitude is a median. Prove that the triangle is isosceles.

78. On each side of an equilateral triangle ABC, congruent segments AB', BC', and AC' are marked, and the points A', B', and C' are connected by lines. Prove that the triangle $A'B'C'$ is also equilateral.

79. Suppose that an angle, its bisector, and one side of this angle in one triangle are respectively congruent to an angle, its bisector, and one side of this angle in another triangle. Prove that such triangles are congruent.

80. Prove that if two sides and the median drawn to the first of them in one triangle are respectively congruent to two sides and the median drawn to the first of them in another triangle, then such triangles are congruent.

81. Give an example of two non-congruent triangles such that two sides and one angle of one triangle are respectively congruent to two sides and one angle of the other triangle.

82.* On one side of an angle A, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ meet on the bisector of the angle A.